

# Modified Laplace transformation method and its application to the anharmonic oscillator

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## Abstract

We apply a recently proposed approximation method to the evaluation of non-Gaussian integral and anharmonic oscillator. The method makes use of the truncated perturbation series by recasting it via the modified Laplace integral representation. The modification of the Laplace transformation is such that the upper limit of integration is cut off and an extra term is added for the compensation. For the non-Gaussian integral, we find that the perturbation series can give accurate result and the obtained approximation converges to the exact result in the  $N \rightarrow \infty$  limit ( $N$  denotes the order of perturbation expansion). In the case of anharmonic oscillator, we show that several order result yields good approximation of the ground state energy over the entire parameter space. The large order aspect is also investigated for the anharmonic oscillator.

## 1 Introduction

Anharmonic oscillator is a system that is well understood in both perturbative and non-perturbative aspects<sup>1,2</sup>. Still, the system plays an important role because it provides a good laboratory for the examination of any calculational scheme newly proposed. The Lagrangian of anharmonic oscillator is given by

$$L = \frac{1}{2} \left( \frac{dq}{dt} \right)^2 - \frac{1}{2} m^2 q^2 - \lambda q^4, \quad (1)$$

and the non-linearity is governed by the coupling constant  $\lambda$ . The perturbation expansion is given in powers of  $\lambda/m^3$ . For instance the ground state energy reads,

$$E = m \sum_{n=0}^{\infty} A_n \left( \frac{\lambda}{m^3} \right)^n, \quad (2)$$

where the coefficients are found as<sup>1</sup>

$$A_0 = \frac{1}{2}, \quad A_1 = \frac{3}{4}, \quad A_2 = -\frac{21}{8} \quad etc. \quad (3)$$

It is known that the coefficient  $A_n$  grows as  $A_n \sim \sqrt{\frac{6}{\pi^3}} (-3)^{n+1} \Gamma(n+1/2)$  for large  $n$  (second paper in ref.1) and thus the series (2) diverges for any small  $\lambda$ . Only for sufficiently small coupling constant,  $\lambda/m^3 \lesssim 0.1$ , the series becomes numerically useful by the appropriate truncation.

The linear  $\delta$  expansion<sup>3</sup> is one of the framework which leads us to go beyond the weak coupling regime. For example, recent studies<sup>4,5</sup> succeeded to approximate the ground state energy in the pure anharmonic case ( $m = 0$ ) by using the perturbative series (2). The strong coupling expansion of the ground state energy is also given in the literature<sup>4</sup> (For the convergence issue, see ref.6).

Recently, a new perturbative scheme was proposed in ref.7. The proposed method utilizes the information contained in the perturbation series by making Heaviside transformation<sup>8</sup> with respect to some parameter (mass in ref.7). In the present paper we extend the method and explore how it can be used to obtain non-perturbative result in the two examples, a non-Gaussian integral and the anharmonic oscillator. We try to construct approximants for the

integral and the ground state energy. For example, the approximant of the ground state energy will be given at  $N$ -th perturbative order for  $\lambda = 1$  as

$$E_{approx}(m) = e^{-m^2 x^*} \sum_{n=0}^N \frac{A_n x^{3n/2-1/2}}{\Gamma(3n/2 + 1/2)} + m \sum_{n=0}^N \frac{\gamma(3n/2 + 1/2, m^2 x^*)}{\Gamma(3n/2 + 1/2)} \frac{A_n}{m^{3n}}, \quad (4)$$

where  $\gamma(p, z)$  denotes the incomplete Gamma function defined by

$$\gamma(p, z) = \int_0^z dt e^{-t} t^{p-1}, \quad (5)$$

and  $x^*$  is the "cut-off" parameter to be determined order by order in some manner. By our approach one can approximate the ground state energy well over *the entire region of the mass  $m$* , including the strong coupling limit and the crossover region from the weak to the strong coupling regime. We also note that application to field theories is straightforward. There exists no renormalization problem.

There is a pioneering work by Graffi et al.<sup>9</sup>, where Borel summation method was used to compute the energy levels of the anharmonic oscillator. Our approach is similar to theirs. However, our approach is different from it in the respects that (i) the variable with respect to which the transformation is carried out is not the coupling constant but the mass square <sup>†</sup> and (ii) the Laplace integration is cut off and the extra term is added. We thus change the representation form itself, while ref.8 modified the integrand according to Padè construction to make up the Borel sum. In our case the integrand is just the truncated one.

This paper is organized as follows: In the next section, we first review and then extend the method of ref.6. The extension results in modifying the ordinary Laplace integral representation as to fit the perturbative approach and enables us to deal with the effect of the explicit mass for both large and small  $m^2$ . In section 3 we perform a simple model calculation of non-Gaussian integral by using the method presented in the previous section. We show that the truncated perturbative series re-constructed by the method gives good approximation of the exact integral and converges to the exact answer in the  $N \rightarrow \infty$  limit ( $N$  denotes the order

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<sup>†</sup>The mass square,  $m^2$ , corresponds to  $\lambda^{2/3}$  for the coupling constant. In terms of  $\lambda^{2/3}/m^2$ , the original perturbation series is not a power series. This is not a negligible difference in our scheme. The issue will be discussed in the last section.

of expansion). In section 4 we turn to the anharmonic oscillator. Via perturbation series, the ground state energy is approximately calculated for various  $\lambda/m^3$  to several higher orders. We will show that, already at 5-th order, the error of our approximation is less than 1 percent for any  $\lambda/m^3$ . In section 5 we address to large order aspects of our approach by proceeding to 249-th order. Some discussion and summary of the present work is given in section 5. In appendix we state the relation between our method and the linear  $\delta$  expansion in a particular limit.

## 2 Modified Laplace transformation

A part of the content of this section is same as the corresponding part in ref. 6. However, to make the presentation self-contained, we allow some overlaps with the work.

For a given physical function  $f(\sigma)$ , we consider the Heaviside transform<sup>7</sup> given by the Bromwich integral,

$$\hat{f}(x) = \int_{p-i\infty}^{p+i\infty} \frac{d\sigma}{2\pi i} \exp(\sigma x) \frac{1}{\sigma} f(\sigma). \quad (6)$$

Here the parameter  $p$  represents the location of the vertical contour. Although  $m^2$  corresponds to  $\sigma$  in the cases we work with, there would be other choices in general. The contour of integration should be placed on the right of all the possible poles and the cut of  $f(\sigma)/\sigma$ . Then, if  $x < 0$ , the contour may be closed into the right half circle and  $\hat{f}(x)$  is found to vanish. From  $\hat{f}(x)$  we have  $f(\sigma)$  via the Laplace integral of the second kind,

$$f(\sigma) = \sigma \int_{-\infty}^{\infty} dx \exp(-\sigma x) \hat{f}(x). \quad (7)$$

Since  $\hat{f}(x) = 0$  when  $x < 0$ , the integration range reduces to  $[0, \infty)$ . However it is convenient to keep the range as  $(-\infty, +\infty)$  to handle partial integration easily.

Now suppose that one is interested in the value of  $f$  at  $\sigma = 0$ , but the perturbative expansion of  $f(\sigma)$  does not allow one to let  $\sigma$  arbitrary small. As in the Fourier transformation, small  $\sigma$  behavior of  $f(\sigma)$  is connected with the large  $x$  behavior of  $\hat{f}(x)$ . More precisely we find that

$$\lim_{\sigma \rightarrow +0} f(\sigma) = \lim_{x \rightarrow \infty} \hat{f}(x), \quad (8)$$

where the existence of both limits are assumed. Our approximation procedure is based on (8) and it goes as follows. Let  $f_N(\sigma)$  denotes the perturbative expansion of  $f(\sigma)$  to  $N$ -th order. Then the corresponding Heaviside function is given by

$$\hat{f}_N(x) = \int_{p-i\infty}^{p+i\infty} \frac{d\sigma}{2\pi i} \frac{\exp(\sigma x)}{\sigma} f_N(\sigma). \quad (9)$$

In cases of our interest, we can not take the naive limits,  $\sigma \rightarrow 0$  or  $x \rightarrow \infty$ , in the both functions. Then there would be two routes to approximate  $f(0)$ . Naive one is to approximate it by fixing  $\sigma$  as small as possible in some manner. The other is, relying upon (8), to approximate  $f(0)$  by  $\hat{f}_N(x)$  where  $x$  should be fixed at some large value,  $x^*$ . Here note that  $\hat{f}(x)$  often has larger convergence radius than  $f(\sigma)$ . For example, if  $f(\sigma) = \sum_{n=0}^{\infty} a_n/\sigma^n$ , then  $\hat{f}(x) = \sum_{n=0}^{\infty} a_n x^n/n!$  ( $x > 0$ ). When  $\hat{f}(x)$  has the convergence radius larger than that of  $f(\sigma)$ , it would be convenient to deal with  $\hat{f}_N$  rather than  $f_N$ . This is because we can probe the large  $x$  behavior of  $\hat{f}$  by  $\hat{f}_N$  so that we have more chance to know the accurate value of  $\hat{f}(\infty)$  and  $f(0)$  accordingly. Therefore, we choose to approximate  $f(0)$  by  $\hat{f}_N(x^*)$ . The explicit way of fixing  $x^*$  is discussed in the next section. Here we just mention that, from the estimation of the upper bound of reliable perturbative region,  $x^*$  will be fixed by the stationarity condition,

$$\left. \frac{\partial \hat{f}_N(x)}{\partial x} \right|_{x=x^*} = 0. \quad (10)$$

If there are several solutions, we should input the largest  $x^*$  into  $\hat{f}_N$ . This is obvious because the value of  $\hat{f}$  at  $x = \infty$  is what we are looking for.

The above approach can be extended to the approximation of the function itself over the entire region of  $\sigma$ . Let us start the discussion by showing how we can approximate the small  $\sigma$  expansion of  $f(\sigma)$ ,

$$f(\sigma) = f(0) + f^{(1)}(0)\sigma + f^{(2)}(0)\frac{\sigma^2}{2!} + \dots \quad (11)$$

As well as  $f(0)$ , we can approximate  $f^{(k)}(0)$  as the following manner: From the formulas,

$$\begin{aligned} \sigma \frac{\partial f(\sigma)}{\partial \sigma} &\rightarrow -x \frac{\partial \hat{f}(x)}{\partial x} \\ \frac{1}{\sigma} f(\sigma) &\rightarrow \int_{-\infty}^x dy \hat{f}(y), \end{aligned} \quad (12)$$

where the rightarrow represents the Heaviside transformation, we have

$$f^{(k)}(\sigma) \rightarrow \int_{-\infty}^x dy (-y)^k \frac{\partial \hat{f}(y)}{\partial y} \stackrel{\text{def}}{=} \alpha_k(x). \quad (13)$$

Assuming the expansion (11), the above two functions agree with each other at  $\sigma = 0$  and  $x = \infty$ . Since what we have at hand is the perturbative one,  $f_N^{(k)}(\sigma)$ , the derivative  $\alpha_k(x)$  is also truncated at order  $N$ . Hence, as in the previous case, we approximate  $f^{(k)}(0)$  by  $\alpha_k(x)$  by fixing the upper limit of integration  $x$  as large as possible within the perturbative region. Replacing  $\hat{f}(y)$  by  $\hat{f}_N(y)$ , we then choose the input  $x$ , say  $x_k^*$ , according to the same logic as that for  $x^*$ . That is,  $x_k^*$  is determined by the stationarity condition,

$$\frac{\partial \alpha_k}{\partial x_k^*} = \frac{\partial}{\partial x_k^*} \int_{-\infty}^{x_k^*} dy (-y)^n \frac{\partial \hat{f}_N(y)}{\partial y} = (-x_k^*)^k \frac{\partial \hat{f}_N(x_k^*)}{\partial x_k^*} = 0. \quad (14)$$

We find that for any  $k$

$$x_k^* = x^* \quad (15)$$

where  $x^*$  is a solution of (10). Substituting  $x^*$  into  $\alpha^{(k)}(x)$  we can construct the approximate Taylor expansion,

$$f(\sigma) \sim \sum_{k=0}^{\infty} \frac{\alpha_k(x^*)}{k!} \sigma^k = \sum_{k=0}^{\infty} \frac{\sigma^k}{k!} \int_{-\infty}^{x^*} dx (-x)^k \frac{\partial \hat{f}_N(x)}{\partial x}, \quad (16)$$

where  $\alpha_0(x^*) = \hat{f}_N(x^*)$ .

Now it is an easy task to obtain expression which can be used for entire  $\sigma$  region: We observe that the right hand side of (16) is easily summed to  $\int_{-\infty}^{x^*} dx e^{-\sigma x} \partial \hat{f} / \partial x$ . Then, by integrating by parts using  $\hat{f}(x) = 0$  for  $x < 0$ , it is written as

$$e^{-\sigma x^*} \hat{f}_N(x^*) + \sigma \int_{-\infty}^{x^*} dx e^{-\sigma x} \hat{f}_N(x) \stackrel{\text{def}}{=} f_N(\sigma, x^*). \quad (17)$$

The left hand side of (17) defines the modification of Laplace transformation. Note that, the naive  $x^* \rightarrow \infty$  limit recovers the ordinary Laplace transform and gives  $f_N(\sigma)$ . We point out, however, that for small  $m$  the dominant contribution comes from the first term. Actually the second term, the cut-off Laplace integral, gives zero in the  $m \rightarrow 0$  limit. Thus the first term is the crucial ingredient in our approach. To summarize, our approach results in approximating  $f(\sigma)$  by  $f_N(\sigma, x^*)$  defined by (17).

It would be interesting to see how the approximant,  $f_N(\sigma, x^*)$ , is different from the ordinary perturbative series. Let us rewrite the integral in (17) as

$$\sigma \int_{-\infty}^{x^*} dx e^{-\sigma x} \hat{f}_N(x) = f_N(\sigma) - \sigma \int_{x^*}^{\infty} dx e^{-\sigma x} \hat{f}_N(x). \quad (18)$$

Then we have

$$f_N(\sigma, x^*) = f_N(\sigma) + f_N^{corr}(\sigma, x^*), \quad (19)$$

where

$$f_N^{corr}(\sigma, x^*) = e^{-\sigma x^*} \hat{f}_N(x^*) - \sigma \int_{x^*}^{\infty} dx e^{-\sigma x} \hat{f}_N(x). \quad (20)$$

If  $f_N(\sigma)$  is given as  $f_N = \sum_{n=0}^N a_n / \sigma^n$ , then  $f_N^{corr}$  is given by

$$f_N^{corr} = e^{-\sigma x^*} \sum_{n=0}^N \frac{a_n x^{*n}}{n!} - \sum_{n=0}^N \frac{a_n \Gamma(n+1, \sigma x^*)}{n! \sigma^n}, \quad (21)$$

where

$$\Gamma(z, p) = \int_p^{\infty} dt e^{-t} t^{z-1}. \quad (22)$$

Using the asymptotic expansion of  $\Gamma(z, p)$ ,

$$\Gamma(z, p) = p^{z-1} e^{-p} \left[ 1 + \sum_{k=1}^{\infty} \frac{1}{p^k} (z-1)(z-2) \cdots (z-k) \right], \quad (23)$$

we find

$$f_N^{corr} = -e^{-\sigma x^*} \sum_{k=1}^{\infty} \frac{1}{\sigma^k} \frac{\partial^k \hat{f}_N(x^*)}{\partial x^{*k}}. \quad (24)$$

From its structure, (24) gives the large  $\sigma$  expansion of  $f_N^{corr}$ . We find that under the stationarity condition (10) the first term in (24) vanishes. This shows that, for large  $\sigma$ , the condition minimizes the deviation of the approximant from the ordinary perturbative result.

### 3 A simple model calculation: non-Gaussian integral

It would be worthwhile carrying out the calculation for a solvable model to gain some concrete feeling on the method to be used. We take up here a non-Gaussian integral,

$$Z(m, \lambda) = \int_{-\infty}^{\infty} dq e^{-m^2 q^2 - \lambda q^4}. \quad (25)$$

The perturbation series is given by expanding the integral in terms of  $\lambda$ . The result reads

$$Z = \sum_{n=0}^{\infty} \frac{(-\lambda)^n}{n!} \frac{\Gamma(2n+1/2)}{m^{4n+1}}. \quad (26)$$

It is easy to see that the series diverges for any  $\lambda/m^4$ . Only when  $\lambda/m^4 \lesssim 0.1$ , the series can be put into numerical use by the appropriate truncation. Nevertheless, we will show that we can obtain the strong coupling ( $\lambda/m^4 \gg 1$ ) expansion from the divergent weak coupling series, and even an approximant which is effective over the entire coupling regime.

First step is to obtain the integrand of Laplace representation. This is done by calculating the integration (6) over  $m^2$  (i.e.,  $\sigma = m^2$ ). Then the result is given by

$$\hat{Z}(x, \lambda) = \sum_{n=0}^{\infty} \frac{(-\lambda)^n}{n!} \frac{x^{2n+1/2}}{(2n+1/2)} \theta(x) = \lambda^{-1/4} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \frac{(\sqrt{\lambda}x)^{2n+1/2}}{(2n+1/2)} \theta(x), \quad (27)$$

where  $\theta(x)$  denotes the ordinary step function ( $\theta(x) = 1$  for  $x > 0$  and 0 for  $x < 0$ ). Note that  $\hat{Z}$  converges for any  $\lambda x^2$ . For the sake of notational simplicity we set  $\lambda = 1$  hereafter.

Let us introduce the perturbatively truncated  $\hat{Z}$  by

$$\hat{Z}_N(x, \lambda = 1) = \hat{Z}_N(x) = \sum_{n=0}^N \frac{(-1)^n}{n!} \frac{x^{2n+1/2}}{(2n+1/2)} \theta(x). \quad (28)$$

Now, we turn to the approximation of  $Z(m, \lambda = 1) = Z(m)$  by using (28). We first discuss how to choose the input  $x^*$ . The break down of perturbative series would generally appear as the rapid rise or fall of the Heaviside function due to the domination of the highest term in (28). Actually, the graphs of the function  $\hat{Z}_N(x)$  to the first several orders show it is the case (see Fig.1). Then, since  $\hat{Z}_N(x)$  is an alternative series, the rise and fall occurs alternatively. For example, for odd  $N$ ,  $\hat{Z}_N(x)$  temporally increases with  $x$  in the reliable perturbative region, but after that (eventually?) it falls down to  $-\infty$ . We note that  $\hat{Z}_N$  behaves temporarily flat just before the break down as shown in Fig.1. In this case the limit of perturbation would emerge typically as the stationary point. This is the reason we employ the stationarity condition,

$$0 = \frac{\partial \hat{Z}_N(x)}{\partial x} = \sum_{n=0}^N \frac{(-1)^n}{n!} x^{2n-1/2} \theta(x) + \sum_{n=0}^N \frac{(-1)^n}{n!} \frac{x^{2n+1/2}}{(2n+1/2)} \delta(x), \quad (29)$$

to fix the perturbative limit,  $x^*$  ( $\delta(x)$  denotes the Dirac delta function). Dropping  $\theta(x)$  and

$\delta(x)$  which are irrelevant, we have

$$\sum_{n=0}^N \frac{(-1)^n}{n!} x^{2n-1/2} = 0. \quad (30)$$

The solution of (30) exists for odd  $N$  and depends on  $N$ . By substituting  $x^*$  into  $\hat{Z}_N(x)$  we have the approximant of  $Z(0)$ . The obtained result is satisfactory as shown in Table 1. Note that  $x^*$  increases with  $N$ , which is a desirable result.

We can also approximate the small mass (strong coupling) expansion of  $Z(m)$ : First note that the coefficients of the  $k$ -th power of  $m^2$ ,  $\alpha_k(x^*)$ , is given by

$$\alpha_k(x^*) = \int_{-\infty}^{x^*} dx (-x)^k \frac{\partial \hat{Z}_N(x)}{\partial x} = (-1)^k \sum_{n=0}^N \frac{(-1)^n (x^*)^{k+2n+1/2}}{n!(k+2n+1/2)}. \quad (31)$$

Thus, at 15-th order for example, we have

$$\begin{aligned} Z_N(m, x^*) &= \sum_{k=0}^{\infty} \frac{\alpha_k}{k!} m^{2k} \\ &= 1.811655 - 0.609988m^2 + 0.223363m^4 - 0.074001m^6 + 0.022042m^8 + \dots \end{aligned} \quad (32)$$

The exact result reads from (25) that  $Z(m) = \frac{1}{2} \sum_{n=0}^{\infty} \Gamma(n/2 + 1/4) (-m^2)^n / n!$  and is given numerically as

$$Z(m) = 1.812805 - 0.612708m^2 + 0.226601m^4 - 0.076589m^6 + 0.023604m^8 + \dots \quad (33)$$

We see the good agreement of (32) and (33) up to several orders. Thus, we reach sufficient accuracy of strong coupling expansion from only the information of the truncated weak coupling series.

Next we examine the approximation of  $Z(m)$  for various  $m$ . The approximant is given from (28) and (17) as,

$$Z_N(m, x^*) = e^{-m^2 x^{*2}} \sum_{n=0}^N \frac{(-1)^n}{n!} \frac{(x^*)^{2n+1/2}}{(2n+1/2)} + \sum_{n=0}^N \frac{(-1)^n}{n!(2n+1/2)} \frac{\gamma(2n+1/2, m^2 x^{*2})}{m^{4n+1}}. \quad (34)$$

Table 2 shows the value of the approximant at  $N = 15$  together with the exact results. We find that the approximant gives good values for the sample of  $m$ .

Before closing this section we show that the approximant  $\hat{Z}_N(x^*)$  converges to  $\hat{Z}(\infty)$  in the  $N \rightarrow \infty$  limit. First we show the convergence of  $\lim_{x \rightarrow \infty} \hat{Z}(x)$ , where  $\hat{Z}(x)$  denotes the exact

transformed function, by calculating Bromwich integral exactly. From (6) and (25)  $\hat{Z}(x)$  is given as

$$\hat{Z}(x) = 2 \int_0^{\sqrt{x}} dq e^{-q^4} = \frac{1}{2} \gamma(1/4, x^2), \quad (35)$$

where we used that  $e^{-m^2 q^2}$  transforms to  $\theta(x - q^2)$ . Thus it is apparent that  $\lim_{x \rightarrow \infty} \hat{Z}(x) = \Gamma(1/4)/2$ , which agrees with  $Z(0)$ .

For the proof we need to know how  $x^*$  behaves for large  $N$ . The relation is found as follows: We note that the condition (30) is viewed as the truncation of the equation,  $x^{-1/2} \exp(-x^2) = 0$ . Since the series expansion of  $\exp(-x)$  has infinitely large convergence radius, the obtained solution tends to  $+\infty$  as  $N \rightarrow \infty$ . More precisely, by assuming the form,  $x^{*2} \sim aN^b$  ( $a, b$ : constant), we find the following scaling at large  $N$ ,

$$x^{*2} \sim \frac{1}{3} N^1. \quad (36)$$

Now, let us define the reminder,  $\hat{R}_N$ , by

$$\hat{R}_N(x) = \hat{Z}_\infty(x) - \hat{Z}_N(x) = \sum_{n=N+1}^{\infty} \frac{(-1)^n}{n!} \frac{x^{2n+1/2}}{(2n+1/2)}. \quad (37)$$

Since  $\hat{Z}_\infty(x)$ , the perturbative series to all orders, apparently converges to exact  $\hat{Z}(\infty)$  because of the infinite convergence radius, it is sufficient to show that

$$\lim_{N \rightarrow \infty} \hat{R}_N(x^*) = 0. \quad (38)$$

This is easily verified: Using the Stirling's formula, we obtain

$$|\hat{R}_N(x^*)| < \sum_{n=N+1}^{\infty} \frac{e^n (x^{*2})^{n+1/4}}{2\sqrt{2\pi} N^{n+3/2}}, \quad (39)$$

and from (36) we then find

$$|\hat{R}_N(x^*)| < \frac{(e/3)^{5/4}}{\sqrt{8\pi}(1-e/3)} N^{-5/4} \left(\frac{e}{3}\right)^N \rightarrow 0 \quad (N \rightarrow \infty), \quad (40)$$

which proves (38).

## 4 Approximation of the ground state energy of the anharmonic oscillator

We turn to discuss the anharmonic oscillator from this section. In complicated systems it is generally hard to proceed to arbitrary higher orders. Hence it is practically important to study whether an employed approximation scheme works at low orders.

In this section we use perturbative series up to 9-th order and show that the several low order result can yield good approximation of the ground state energy via modified Laplace representation.

Let  $\lambda = 1$  for notational simplicity. Our first task is to obtain the Heaviside function of the perturbative ground state energy,  $E_N(m)$ ,

$$E_N(m) = \sum_{n=0}^N \frac{A_n}{(m^2)^{3n/2-1/2}}. \quad (41)$$

Heaviside transform of  $m^{-3n+1}$  with respect to  $m^2$  gives

$$m^{-3n+1} \rightarrow \frac{x^{3n/2-1/2}}{\Gamma(3n/2 + 1/2)} \theta(x). \quad (42)$$

Then from (41) and (42) we have the Heaviside function,  $\hat{E}_N(x)$ ,

$$\hat{E}_N(x) = \sum_{n=0}^N \frac{A_n x^{3n/2-1/2}}{\Gamma(3n/2 + 1/2)} \theta(x). \quad (43)$$

For notational simplicity we omit the step function in what follows.

By suitable replacement of variable, the function  $\hat{E}$  agrees with the function appeared in ref. 5 in which the  $\delta$  expansion method was applied to the anharmonic oscillator (see eq.(7) of ref. 5). Authors of ref. 5 considered the case where the coupling,  $\delta\Omega^2$ , grows to large orders ( $\delta$  represents a fictitious parameter which is to be set 1 at the end of calculation and  $\Omega^2$  denotes the mass auxiliary introduced to divide the given Lagrangian un-conventionally). Then the authors found that their approximant converges to the form (43) in that large order limit <sup>‡</sup>.

<sup>‡</sup>. This implies that there may be some connection between the linear  $\delta$  expansion and our

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<sup>‡</sup>In  $\delta$  expansion scheme, this limit corresponds to infinite perturbative order. Then we note that in this limit the parameter  $\Omega$  is no longer arbitrary because it should be of the same order of magnitude with the perturbative order. We also point out that this limit was considered in ref.9 from different motivation.

method. Actually we show in appendix that it is the case; We will show that, in a particular limit, the Heaviside transform is induced in a suitable interpretation.

Now, let us discuss the energy approximation using (43). As in the previous section we use the following condition,

$$\frac{\partial \hat{E}_N(x)}{\partial x} = 0, \quad (44)$$

to fix the input  $x^*$ . Up to  $N = 9$ , while there is no solution for even  $N$ , we find just one solution for each odd  $N$ . By substituting  $x^*$  into  $\hat{E}_N(x)$ , we obtain the approximation of  $E(0)$  which is the energy in the pure anharmonic case or in other words in the strong coupling limit. We also evaluate the succeeding Taylor coefficients of first five terms. For the purpose it may be convenient to use the integrated form of  $\alpha_k$ ,

$$E^{(k)}(0) \sim \alpha_k = (-1)^k \sum_{n=0}^N \frac{A_n (3n/2 - 1/2) (x^*)^{k+3n/2-1/2}}{\Gamma(3n/2 + 1/2)(k + 3n/2 - 1/2)} \quad (k = 1, 2, 3, \dots), \quad (45)$$

which comes from substitution of (43) into the general formula (13). Then, we have the strong coupling expansion,

$$E(m) \sim \alpha_0 + \frac{\alpha_1}{1!} m^2 + \frac{\alpha_2}{2!} m^4 + \dots, \quad (46)$$

where  $\alpha_0 = \hat{E}_N(x^*)$ . We have done numerical calculation with Mathematica. As shown in Table 3, the obtained results agree well with the recent result reported by Kleinert<sup>4</sup>. Next we check the approximation for various  $m$ . For the purpose we substitute (43) into (17) and use the definition of incomplete Gamma function (5) to obtain the approximant,  $E_N(m, x^*)$ . The result reads at the order  $N$  as

$$E_N(m, x^*) = e^{-m^2 x^*} \sum_{n=0}^N \frac{A_n x^{3n/2-1/2}}{\Gamma(3n/2 + 1/2)} + m \sum_{n=0}^N \frac{\gamma(3n/2 + 1/2, m^2 x^*)}{\Gamma(3n/2 + 1/2)} \frac{A_n}{m^{3n}}. \quad (47)$$

We find that the numerical calculation shows the good results for all  $m$  <sup>§</sup>. From these results, we find that the accuracy of calculated energy is quite satisfactory. Already at the 1-st order, the approximation gives error only within 11 percent for all  $m$ . And at 5-th order, the error is less than 1 percent. These results are depicted in Fig.2. In particular we note that  $E_N(m, x^*)$

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<sup>§</sup>The reference value is generated according to the method of ref.11. We thank H. Suzuki for the correspondence of the reference.

improves the approximation of  $E$  in the crossover region of weak and strong coupling regimes,  $0.1 \lesssim 1/m^2 \lesssim 1$ . Note that in this intermediate coupling region the ordinary perturbation series can not be used because the series never be close to the exact value under any truncation.

It would be interesting to see the explicit difference of our approximant from the ordinary perturbative series. Repeating the steps from (18) to (24), we find

$$E_N(m, x^*) = E_N(m) + E_N^{corr}(m, x^*), \quad (48)$$

where

$$E_N^{corr}(m, x^*) = e^{-m^2 x^*} \hat{E}_N(x^*) - m \sum_{n=0}^N \frac{\Gamma(3n/2 + 1/2, m^2 x^*)}{\Gamma(3n/2 + 1/2)} \frac{A_n}{m^{3n}} = -e^{-m^2 x^*} \sum_{i=1}^{\infty} \frac{b_{i,N}}{(m^2)^i}, \quad (49)$$

and

$$b_{i,N} = \sum_{n=0}^N \frac{A_n(x^*)^{3n/2-1/2-i}}{\Gamma(3n/2 + 1/2)} \prod_{j=1}^i (3n/2 + 1/2 - j) = \frac{\partial^i \hat{E}_N(x^*)}{\partial x^{*i}}. \quad (50)$$

The coefficients slowly changes order by order except for  $b_{1,N}$  which satisfies from (44),

$$b_{1,N} = 0. \quad (51)$$

We have done numerical computation of  $b_{i,N}$  at  $N = 1, 3, 5, 7, 9$ . To several  $i$ , we find that the size of  $b_{i,N}$  decreases as  $N$  increases. This is an expected tendency. In fact, we can show that  $b_{i,N}$  should converges to zero in the  $N \rightarrow \infty$  limit. The basic relation to be noted is that

$$\sigma f(\sigma) \rightarrow \frac{\partial \hat{f}(x)}{\partial x}. \quad (52)$$

From (52), it is easy to see that

$$\sigma^i f(\sigma) \rightarrow \frac{\partial^i \hat{f}(x)}{\partial x^i}, \quad (53)$$

and for our case,

$$m^{2i} E(m) \rightarrow \frac{\partial^i \hat{E}(x)}{\partial x^i}. \quad (54)$$

There should be the agreement condition (8) between these functions and therefore, noting that  $\lim_{m^2 \rightarrow 0} m^{2i} E(m) = 0$ , the coefficients  $b_{i,N}$  should tend to zero as  $N$  increases to infinity if the approximation procedure is working well. This issue will be studied further in the next section.

## 5 Higher order behavior

The important information in our approach is contained in the Heaviside function  $\hat{E}(x)$ . We therefore investigate the properties of  $\hat{E}(x)$  relevant to our analysis by extending the perturbative order up to 249-th.

An important issue in our approach is whether  $\lim_{x \rightarrow \infty} \hat{E}(x)$  exists or not. Although we do not have rigorous proof, we see convincing answer by figuring out  $\hat{E}_N(x)$  to large  $N$ . We have generated perturbative coefficients  $A_n$  up to 249 terms with the help of Mathematica and plotted the graph of  $\hat{E}_{249}(x)$  as shown in Fig.3. We note that plateau starts around  $x \sim 1$  and abruptly grows up around  $x \sim 3.2$ , which shows the break down of perturbation expansion. Taking closer look, we find that the function  $\hat{E}_{249}(x)$  weakly oscillates at the plateau region. The amplitude of the oscillation is very tiny indeed; The difference between the first extremum and the next is just  $0.0000103\cdots$  which should be compared with the first extremum value,  $0.667975902279\cdots$ . The difference between the second and third is about  $0.00000001$ . Thus the amplitude decreases as the function oscillates to larger  $x$ . The values of three stationary points of  $\hat{E}_{249}$  are given as

$$\begin{aligned}\hat{E}_{249} &= 0.667975902279\cdots, & x &= 1.139689002700 \\ \hat{E}_{249} &= 0.667986268727\cdots, & x &= 2.069065340532 \\ \hat{E}_{249} &= 0.667986259143\cdots, & x &= 2.987637042160.\end{aligned}\tag{55}$$

These values shows how the  $\hat{E}_N$  at the plateau is close to the value  $E(0)$  which is known to be<sup>12</sup>

$$E(0) = 0.667986259155777108270962\cdots\tag{56}$$

Thus the behavior of  $\hat{E}_{249}(x)$  for  $x \leq 3$ , where the function is reliable, strongly suggests that  $\lim_{x \rightarrow \infty} \hat{E}(x)$  would exist and consequently agree with  $E(0)$ .

The first, second and third stationary points begins to appear from 28,101 and 246 orders, respectively. Hence, several solutions exist for some orders higher than 28-th. This phenomenon was also observed in the  $\delta$  expansion framework<sup>5</sup>. There it was found that the value

of interest at largest  $1/\Omega$  was most accurate. But within the framework there is no a priori reason why one should take the largest  $1/\Omega$ . On the other hand, it is obvious in our approach that one should focus on the largest  $x^*$  as we mentioned in section 2. The behavior of our approximants  $\hat{E}_N(x^*)$  as the order increases is as follows. For  $x$  smaller than the largest  $x^*$ ,  $\hat{E}_N(x)$  is a good approximation of the exact function and the departure starts around  $x \sim x^*$ . Therefore the largest stationary point sits in the vicinity of the exact function, and slides to larger  $x$  direction along the curve of the exact function as  $N$  increases. Hence the convergence issue of the approximants,  $\{\hat{E}_N(x^*)|N = 1, 2, \dots\}$ , is tightly connected with the convergence of  $\lim_{x \rightarrow \infty} \hat{E}(x)$  and how the stationary solution,  $x^*$ , grows with the order. Since, by the definition, the largest  $x^*$  is located at the upper limit of reliable region,  $\{\hat{E}_N(x^*)|N = 1, 2, \dots\}$  would converge to  $\hat{E}(\infty)$ . It is also clear that  $\hat{E}_N(x^*)$  oscillates as  $N$  increases by following the function  $\hat{E}(x)$ . This oscillatory property of the approximant was observed (but not clarified) by Kleinert<sup>4</sup>.

As is obvious from the above discussion, the largest stationary point around the order  $N = 246$  (at which the third stationary point of  $\hat{E}(x)$  is settled) is approximately given by the third stationary point shown in the last of (55). Therefore using that value of  $x^*$ , we can see how accurate  $E_N(m, x^*)$  is for various  $m^2$ . The result of computer calculation is shown in Table 4 and shows that the obtained values are quite accurate for all  $m^2$ .

Finally let us comment on the behavior of  $b_{i,N}$ , the coefficients of  $E_N^{corr}$  at large  $m$ , for large  $N$ . We have calculated them at  $N = 28, 101, 249$ . We find that the size of  $b_{i,N}$  decreases to zero as the order  $N$  increases. For example, the results for  $i = 1$  to  $7$  at  $N = 249$  are respectively given as follows;

$$\begin{aligned} b_{1,249} &= 0\text{E}-22, \quad b_{2,249} = 8.259931\text{E}-10, \quad b_{3,249} = -8.27746\text{E}-9, \quad b_{4,249} = 5.094257\text{E}-7, \\ b_{5,249} &= 4.804239\text{E}-5, \quad b_{6,249} = 6.054357\text{E}-3, \quad b_{7,249} = 7.451039\text{E}-1, \end{aligned} \quad (57)$$

Thus, for lower  $i$ , the result almost agrees with the requirement that  $\lim_{x \rightarrow \infty} \partial^i \hat{E}(x) / \partial x^i = 0$ . For larger  $i$ , however, the coefficient  $b_{i,249}$  grows rapidly. This represents that as  $x$  exceeds  $x^*$   $\hat{E}_{249}(x)$  abruptly grows as shown in Fig.3.

## 6 Discussion and conclusion

Critical things leading our scheme to the success are that the convergence radius,  $\rho$ , is infinite for the Heaviside functions  $\hat{f}(x)$  and that they quickly approach to the value  $\hat{f}(\infty)$  at finite  $x$ . The later fact is confirmed numerically for the anharmonic oscillator from the remarkable closeness of  $\hat{E}_N(x)$  at  $x \in [1, 3]$  to  $\hat{E}(\infty)$ . For the non-Gaussian integral it is analytically clarified as follows: By using  $\gamma(1/4, x^2) = \Gamma(1/4) - \Gamma(1/4, x^2)$  and the expansion (23) we obtain from (35) that

$$\hat{Z}(x) = \frac{1}{2} \left[ \Gamma(1/4) - x^{-3/2} \exp(-x^2) \left( 1 + \sum_{i=1}^{\infty} x^{-i} \prod_{j=1}^i (1/4 - j) \right) \right]. \quad (58)$$

The approach to  $\hat{f}(\infty)$  is speedy because of exponential damping at large  $x$ . It is interesting to note that  $\hat{Z}(x)$  has no power-like term,  $x^{-n}$ , in large  $x$  expansion (58).

Actually these features are related with the choice of integration variable generally denoted as  $\sigma$ . We discuss on this issue by our two examples.

First consider the non-Gaussian integral case and let  $\sigma = m^\beta$ . We numerically study on how the function  $\hat{Z}$  varies according to the power  $\beta$ . In calculating the transform, the following result is convenient to use:

$$\sigma^\xi \rightarrow \frac{x^{-\xi}}{\Gamma(1 - \xi)}, \quad (59)$$

where we have dropped possible  $\theta$  and  $\delta$  functions. From (59) it is easy to find that, with respect to  $m^\beta$ , the transformed series is given by

$$\hat{Z}_N(x) = \sum_{n=0}^N (-1)^n \frac{\Gamma(2n + 1/2)}{n! \Gamma(4n/\beta + 1/\beta + 1)} x^{4n/\beta + 1/\beta}. \quad (60)$$

Then we find that  $\hat{Z}_\infty(x)$  is a divergent series for  $\beta > 4$  and becomes a convergent one for  $\beta = 4$  where the convergence radius is  $1/4$ . When  $\beta < 4$ , the series becomes a convergent one for any  $x$ . We have plotted in Fig.4 the series for  $\beta \in [1, 3]$  at  $N = 100$ . Next problem is whether the limit  $\lim_{x \rightarrow \infty} \hat{Z}_\infty(x)$  exists or not. From the numerical calculation at  $N = 200$ , we find that the result changes around  $\beta \sim 1.32$ : For  $\beta \in [1, 1.32]$ ,  $\hat{Z}_\infty(x)$  would not converge in the  $x \rightarrow \infty$  limit. The convergence of  $\lim_{x \rightarrow \infty} \hat{Z}_\infty(x)$  seems to be realized for  $\beta \in [1.32, 4]$ .

We do not know why around  $\beta = 1.32$  the convergence property changes. Now, we observe the following features: When  $\beta$  is small the truncated series becomes effective to large  $x$  but  $\hat{Z}(x)$  does not converge at  $x = \infty$ . As  $\beta$  increases beyond  $\sim 1.32$  but still within  $\beta < 4$ , the effective range of truncated series decreases but the saturation to finite  $\hat{Z}(\infty)$  is fast. Furthermore the oscillation amplitude damps as  $\beta$  decreases. From these points, it is subtle that, at a fixed order, what value of  $\beta$  between  $\sim 1.32$  and 4 gives the best approximation. We have calculated  $\hat{Z}_N(x^*)$  at  $N = 99$  or 100 for  $\beta = 1.5, 1.7, 1.9, 2.0$  and 2.1. The result reads

$$\begin{aligned}
\beta = 1.5 & \quad 1.8169575 \quad (x = 15.037, \ N = 100) \\
\beta = 1.7 & \quad 1.81926459 \quad (x = 10.336, \ N = 100) \\
\beta = 1.9 & \quad 1.81719914639 \quad (x = 7.1806, \ N = 100) \\
\beta = 2.0 & \quad 1.812804954110934 \quad (x = 5.34, \ N = 99) \\
\beta = 2.1 & \quad 1.8051679 \quad (x = 5.043, \ N = 99). \tag{61}
\end{aligned}$$

Thus we find that the best approximation is realized at  $\beta = 2.0$ . The agreement to the exact value,  $1.812804954110954\cdots$  is remarkable for that case (The first 14 decimals are in the agreement).

We note that the ordinary Borel summation method employees  $\lambda$  as the transformation variable. This means in our context that  $\sigma = m^4$ . Then the convergence radius of perturbative  $\hat{Z}(x)$  becomes finite ( $\rho = 1/4$ ) and the choice is not suited for our modified Laplace representation approach where the integrand is truncated and the upper integration limit is cut off.

Now we turn to the anharmonic oscillator. If one performs Heaviside transformation with respect to  $m^\beta$ , one has

$$\hat{E}_N(x) = \sum_{n=0}^N \frac{A_n}{\Gamma(3n/\beta - 1/\beta + 1)} x^{(3n-1)/\beta}. \tag{62}$$

The  $n$ -th coefficient behaves for large  $n$  as

$$\frac{A_n}{\Gamma(3n/\beta - 1/\beta + 1)} \sim (-1)^{n+1} \left(\frac{3n}{e}\right)^{(1-3/\beta)n} \beta^{(3n-1)/\beta} n^{-1/2+1/\beta}. \tag{63}$$

Hence the series is divergent for  $\beta > 3$  and  $\rho$  is infinite for  $\beta < 3$ . For  $\beta = 3$  the series is a convergent one, but  $\rho = 1/3$ . We have figured out  $\hat{E}_{249}(x)$  by varying  $\beta$  from 1 to 3. The result is shown in Fig.5. The convergence property of  $\hat{E}_\infty(x)$  in the  $x \rightarrow \infty$  limit seems to change around  $\beta \sim 1.15$ . As in the previous case, we examined what value around  $\beta = 2$  gives the best approximation of  $E(0)$  at  $N = 248$  or 249. For  $\beta = 1.7, 1.8, 1.9, 2.0, 2.1$  we obtained

$$\begin{aligned}
\beta = 1.7 & \quad 0.6655 \quad (x = 6.53 \sim 6.57, N = 248) \\
\beta = 1.8 & \quad 0.665514724 \quad (x = 5.1257 \sim 5.1260, N = 248) \\
\beta = 1.9 & \quad 0.666234824085 \quad (x = 4.0185, N = 248) \\
\beta = 2.0 & \quad 0.667986259143255939 \quad (x = 2.98685, N = 248) \\
\beta = 2.1 & \quad 0.67107970759 \quad (x = 2.5012, N = 249).
\end{aligned} \tag{64}$$

From the above sample, we find that the choice  $\beta = 2$  gives the particular good approximation. The ordinary Borel choice of variable,  $\sigma = m^3$ , also does not work well in our approach. This is because in that choice the convergence radius of perturbative  $\hat{E}(x)$  becomes finite ( $\rho = 1/3$ ).

It is interesting to note that why the choice,  $\beta = 2$ , gives the best approximation in both cases. Although we have not resolved it yet, we suspect that the resolution would have related to the fact that  $Z(m)$  and  $E(m)$  has expansion in the square of the mass,  $c_0 + c_1 m^2 + c_2 (m^2)^2 + \dots$ . Any finite sum,  $\sum_{i=0}^n c_i (m^2)^i$ , vanishes, if the integration variable is chosen as  $\sigma = m^2$  because from (59),

$$\sigma^i \rightarrow 0 \quad (i = 1, 2, \dots). \tag{65}$$

Hence, there is no powers of  $1/x$  in the corresponding Heaviside function at large  $x$ . Note that when the power  $\xi$  is fractional the transform of  $\sigma^\xi$  survives to give  $x^{-\xi}$ . Thus, as well as  $\hat{Z}(x)$  which is analytically solved as (58),  $\hat{E}(x)$  would be suppressed exponentially only at  $\beta = 2$  and the smallness of the difference  $|\hat{E}(x) - \hat{E}(\infty)|$  would be thus explained.

To conclude this paper, we have demonstrated how truncated perturbation series can be utilized to approximate the non-perturbative quantities via modified Laplace representation. The non-Gaussian integral is precisely calculated from the truncated perturbation series. And

the approximant for the strong coupling limit is found to converge in the  $N \rightarrow \infty$  limit. The ground state energy of the anharmonic oscillator is also approximated successfully over the entire parameter space. Large order calculation strongly suggests that our approach works well to all orders. In both examples, we find that the perturbative knowledge serves us sufficient information for recovering the small mass or strong coupling expansion.

In field theories, however, perturbation series has more complex structure in general. For example, the perturbation series for the dressed mass, vacuum energy density or condensates involves the mass logarithm,  $\log m$ . When the mass-logs exist, it is no longer expected that the convergence radius is enlarged by Heaviside transformation with respect to the mass, although some qualitative improvement is found in the literature (first paper in ref.6). This is the subject of our future investigation.

## Appendix: Linear $\delta$ expansion in a particular limit and the Heaviside transformation

In this Appendix we show that, in a particular limit, the linear  $\delta$  expansion leads to the Heaviside transformation.

In linear  $\delta$  expansion, the Lagrangian of anharmonic oscillator is written by introducing auxiliary or variational mass  $\Omega$  as,

$$L = \frac{1}{2} \left( \frac{dq}{dt} \right)^2 - \frac{1}{2} \Omega^2 q^2 - \frac{1}{2} (m^2 - \Omega^2) q^2 - \lambda q^4, \quad (66)$$

and the free and interaction parts are defined as

$$\begin{aligned} L_{free} &= \frac{1}{2} \left( \frac{dq}{dt} \right)^2 - \frac{1}{2} \Omega^2 q^2, \\ L_{int} &= \frac{1}{2} \delta \Omega^2 q^2 - \lambda q^4, \end{aligned} \quad (67)$$

where

$$\delta = 1 - m^2/\Omega^2. \quad (68)$$

Let us concentrate on the  $m = 0$  case hereafter. Then one should set  $\delta = 1$ , which represents the masslessness of the starting Lagrangian, at the end of the calculation.

First we point out that perturbation expansion in  $L_{int}$  is generated from the ordinary perturbative result with mass  $\Omega$  by shifting  $\Omega^2 \rightarrow \Omega^2(1 - \delta)$  and then expanding in powers of  $\delta$  to relevant orders. Let the number of  $\delta\Omega^2$  vertex and  $\lambda$  coupling in a given Feynman diagram be  $n_\Omega$  and  $n_\lambda$  respectively. Then at order  $n$  in  $L_{int}$ , any contributing diagram should obey  $n_\Omega + n_\lambda \leq n$ . To proceed to large orders keeping the analytical manipulation straightforward, it is however convenient to modify the expansion scheme. The new expansion is defined such that, at order  $n$ , contributing diagram should obey  $n_\lambda \leq n$  and  $n_\Omega \leq n$ . Even a diagram has  $n$  vertices for  $\lambda$  coupling and  $n$  mass insertions, it should be included to the  $n$ -th order expansion. In this expansion scheme, any contributing diagrams are obtained such that, given a Feynman diagram with no  $\delta\Omega^2$  vertex, one should incorporate it by shifting  $\delta$  and then expanding in  $\delta$  up to just the order of perturbative expansion. This procedure is formally carried out as we can see below.

Consider a given Feynman amplitude with no  $\delta\Omega^2$  vertex,  $f(\Omega^2)$ . First we shift  $\Omega^2 \rightarrow \Omega^2(1 - \delta)$  and expand the result in  $\delta$ . We then have

$$f(\Omega^2 - \Omega^2\delta) = \sum_{k=0}^{\infty} f^{(k)}(\Omega^2) \frac{(-\Omega^2\delta)^k}{k!} = \lim_{N \rightarrow \infty} \sum_{k=0}^N \frac{(-\delta\Omega^2)^k}{k!} \left(\frac{\partial}{\partial\Omega^2}\right)^k f(\Omega^2). \quad (69)$$

Setting  $\delta = 1$  we have

$$f(0) = \lim_{N \rightarrow \infty} \sum_{k=0}^N \frac{(-\Omega^2)^k}{k!} \left(\frac{\partial}{\partial\Omega^2}\right)^k f(\Omega^2). \quad (70)$$

By using

$$\frac{\partial}{\partial p} \frac{1}{p} = \frac{1}{p} \frac{\partial}{\partial p} - \frac{1}{p^2}, \quad (71)$$

we find

$$\sum_{k=0}^N \frac{(-p)^k}{k!} \left(\frac{\partial}{\partial p}\right)^k = p \frac{(-p)^N}{N!} \left(\frac{\partial}{\partial p}\right)^N \frac{1}{p} \stackrel{\text{def}}{=} \mathcal{D}_N(p). \quad (72)$$

Thus we arrive at

$$f(0) = \lim_{N \rightarrow \infty} \mathcal{D}_N(\Omega^2) f(\Omega^2). \quad (73)$$

This is, however, just a formal result and we address explicit example and ask what comes out.

The operation of  $\mathcal{D}_N(\Omega^2)$  on  $(\Omega^2)^\xi$  gives

$$\mathcal{D}_N(\Omega^2)(\Omega^2)^\xi = \frac{(N - \xi) \cdots (2 - \xi)(1 - \xi)}{N!} (\Omega^2)^\xi. \quad (74)$$

When  $N$  is large enough, the right hand side approaches to  $(\Omega^2/N)^\xi / \Gamma(1 - \xi)$  and the result of operation of  $\mathcal{D}_N(\Omega^2)$  on  $E_N(\Omega)$  reads,

$$\mathcal{D}_N(\Omega^2)E_N(\Omega) \sim \sum_{n=0}^N \frac{A_n}{\Gamma(3n/2 + 1/2)} \left(\frac{N}{\Omega^2}\right)^{3n/2-1/2} \quad (N \gg 1). \quad (75)$$

Thus when  $N/\Omega^2$  is changed to  $x$  in (75) the result agrees with (43). However note that, in accord with  $N$ , we must let  $\Omega$  large enough to stop the divergence of RHS of (75) (Note that at  $N = 249$ , the largest stationary point is given at  $N/\Omega^2 \sim 3$ ). To give (75) some meaning, it is thus necessary to let  $N$  and  $\Omega^2$  simultaneously large with  $N/\Omega^2$  kept finite. And accordingly,  $\Omega^2$  is no longer a free finite parameter. If we adopt this recipe, the equality of (73) breaks down and we should take the righthand side of (73) as a new function  $\hat{f}$  of variable  $x = N/\Omega^2$ . We can expect the equality relation only in the limit,  $x \rightarrow \infty$ , such that  $\lim_{x \rightarrow \infty} \hat{f}(x) = f(0)$  (This limit corresponds to  $N \rightarrow \infty$  with  $\Omega^2$  fixed).

Now we show that  $\mathcal{D}_N$  induces Heaviside function in the limit that  $N, \Omega \rightarrow \infty$  with  $\Omega^2/N$  fixed. Let us start with the Laplace transform,

$$f(\Omega) = \Omega^2 \int_{-\infty}^{\infty} dt \exp(-\Omega^2 t) \hat{f}(t). \quad (76)$$

By operating  $\mathcal{D}_N$  we have

$$\mathcal{D}_N f(\Omega) = \int_{-\infty}^{\infty} dt \frac{(\Omega^2)^{N+1} t^N}{N!} \exp(-\Omega^2 t) \hat{f}(t) \stackrel{\text{def}}{=} \int_{-\infty}^{\infty} dt \Delta_{N,\Omega^2}(t) \hat{f}(t). \quad (77)$$

We find that the function  $\Delta_{N,\Omega^2}(t)$  approaches to the Dirac  $\delta$  function in the limit,  $N, \Omega \rightarrow \infty$  with the ratio  $N/\Omega^2$  fixed,

$$\Delta_{N,\Omega^2}(t) \rightarrow \delta(t - N/\Omega^2). \quad (78)$$

Thus we find that

$$\lim_{N,\Omega^2 \rightarrow \infty} \mathcal{D}_N f(\Omega) = \hat{f}(N/\Omega^2). \quad (79)$$

This result states that  $\mathcal{D}_N$  gives the kernel of Laplace transform in the limit  $N, \Omega \rightarrow \infty$  with the ratio  $N/\Omega^2$  fixed.

We remark that although we have shown that, in our modified expansion scheme, the  $\delta$  expansion can induce the Heaviside transform in large orders, this may not lead that the conventional expansion in  $L_{int}$  also gives the same function  $\hat{f}$  in the large order limit. This is because the difference of the included diagrams between the two expansions increases as the order increases.

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### Table Captions

**Table 1** Numerical computation of  $Z_N(x^*)$  to 15-th orders.

**Table 2** Numerical computation of  $Z_{15}(m^2, x^*)$  for various values of  $m^2$ . At this order  $x^{*2} = 5.0438870$ .

**Table 3** Low order approximation of first several strong coupling coefficients of the ground state energy.

**Table 4** Numerical result of  $E_{249}(m^2, x^*)$  for various  $m^2$ .

### Figure Captions

**Fig. 1** The perturbative Heaviside functions  $\hat{Z}_N(x)$  for  $N = 1, 4, 7$ .

**Fig. 2** The ratio,  $E_N(m, x^*)/E_{exact}(m)$  is plotted at  $N = 1$  and  $5$ .

**Fig. 3** The function  $\hat{E}_{249}(x)$  is shown. The sharp rise around  $x \sim 3.2$  represents the break down of the perturbative truncation.

**Fig. 4** The function  $\hat{Z}_{100}(x)$  is shown for various choice of  $\beta$ . The amplitude of oscillation of  $\hat{Z}_{100}(x)$  decreases as  $\beta$  decreases from  $1.0$  to  $2.0$ . The effective range of the series is larger for smaller  $\beta$  but the saturation to  $\hat{Z}(\infty)$  is slower. When  $\beta$  is close to  $2.0$ , the oscillation is weak and the saturation to  $\hat{Z}(\infty)$  is very fast.

**Fig. 5** The function  $\hat{E}_{249}(x)$  is shown for various  $\beta$ . At  $\beta = 1$ ,  $\hat{E}_N(x) \rightarrow 0$  as  $x \rightarrow 0$ . When  $1 < \beta \leq 3$ ,  $\hat{E}_N(x)$  diverges in the  $x \rightarrow 0$  limit. The oscillation amplitude of  $\hat{E}_{249}(x)$  damps as  $\beta$  decreases. At  $\beta = 2.0$ , the oscillation is quite weak and the saturation to  $\hat{E}(\infty)$  is very fast.

$N$	$\hat{Z}_N(x^*)$	$x^{*2}$
1	1.6	1
3	1.7313594	1.5960716
5	1.7765256	2.1806071
7	1.7955618	2.7590027
9	1.8043006	3.3335514
11	1.8085078	3.9054517
13	1.8105959	4.4754119
15	1.8116546	5.0438870
exact	1.8128049	

Table 1

$m^2$	$Z_{15}(m^2, x^*)$	<i>exact</i>
0.01	1.80557702921362946	1.806700454307384679
0.1	1.75281762908207768	1.753725831772014832
1	1.36831695165151724	1.368426855735508774
3	0.96173724333279584	0.961738333157472108
6	0.710038679143677110	0.710038680405132855
10	0.556465718382570615	0.556465718382772753
100	0.177232097497741759	0.177232097497741761

Table 2

$N$	$\hat{E}(x^*)$	$\alpha_1$	$\alpha_2$
1	0.738558766382022	0.121215344755496	-0.004420970641441
3	0.686283726385561	0.134984882799344	-0.006437328047158
5	0.674564660427775	0.139821127686036	-0.007465973798930
7	0.670682699394559	0.141853009994811	-0.008005281363443
9	0.669175108154224	0.142780937393257	-0.008293166786158
exact	0.667986259155...	0.143668783380...	-0.008627565680...

$N$	$\alpha_3$	$\alpha_4$	$x^*$
1	0.0002176764415436	-0.000010207418675	0.328248340614232
3	0.0004280233911678	-0.000027447789756	0.448360373271548
5	0.0005773467460109	-0.000044015663387	0.549152913559036
7	0.0006739106832314	-0.000057114935211	0.636621775859320
9	0.0007338756445212	-0.000066541377300	0.713814879924458
exact	0.000818208905...	-0.000082429217...	

Table 3

$m^2$	$E_{249}(m^2, x^*)$	exact
0.001	0.66812991929974827	0.66812991931241042
0.01	0.66942208503810206	0.66942208505040309
0.1	0.68226767187380087	0.68226767188301217
1	0.80377065123375873	0.80377065123427376
10	1.64938954183035211	1.64938954183035211
100	5.00747395574729234	5.00747395574729234
1000	15.8121382178529	15.8121382178529

Table 4